THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2023-24
Tutorial 6 solutions
19th October 2023

- The tutorial solutions are written for reference and proofs will be sketched briefly. You should try to fill in the details as an exercise. The solutions for Homework optional questions can be found in Homework solutions, which would be released after the deadlines. Please send an email to echlam@math.cuhk.edu.hk if you have any further questions.

1. The action is free $\iff g \cdot x = x$ for some x implies that $g = e \iff$ for any $g \neq e$, $g \cdot x \neq x \iff G_x = \{e\}$ for all $x \in X$. The left regular action of G on itself is both transitive and free. Transitivity: if $h_1, h_2 \in G$ then $h_2 = (h_2 h_1^{-1}) \cdot h_1$. Freeness: if $gh = h$ then $g = e$.

2. Since an $n \times n$ matrix is invertible if and only if its column space has rank n, we can regard an element $[v_1, ..., v_n] \in F(\mathbb{R}^n)$ as an invertible $n \times n$-matrix, so that $F(\mathbb{R}^n)$ is in bijection with $GL(n, \mathbb{R})$ as sets (despite there is no group structure on $F(\mathbb{R}^n)$. This idenitification is equivariant in that the action of $GL(n, \mathbb{R})$ on $F(\mathbb{R}^n)$ is exactly given by matrix multiplication $A \cdot [v_1, ..., v_n] = A[v_1 | \cdots | v_n]$. Given any two ordered basis $[v_1, ..., v_n]$ and $[w_1, ..., w_n]$, denote the corresponding matrices as A, B, then $[w_1, ..., w_n] = BA^{-1}[v_1, ..., v_n]$ implies that the action is free. And transitivity follows from $A[v_1 | \cdots | v_n] = [v_1 | \cdots | v_n] \implies A = I$ by right cancellation. Alternatively, this action is the same as left regular action of $GL(n, \mathbb{R})$ on itself, so by Q1 it is free.

3.
$$\begin{aligned} g \in \ker(\rho) &\Leftrightarrow g \cdot x = x,\ \forall x \in X \\ &\Leftrightarrow g \in G_x,\ \forall x \in X \\ &\Leftrightarrow g \in \bigcap_{x \in X} G_x. \end{aligned}$$

4. (a) Suppose $h \in G_{x'}$, then $hg \cdot x = h \cdot x' = x' = gx$, so $g^{-1}hg \in G_x$, therefore $h \in gG_xg^{-1}$. Since $G \cdot x = G \cdot x'$, we know that the stabilizers have the same cardinality, therefore $G_{x'} = gG_xg^{-1}$.

   (b) For a proper subgroup H of a finite group G, there are at most $[G : H]$ many distinct conjugate subgroups gHg^{-1}. But these groups all share a common identity element, so there are at most $[G : H](|H| - 1) + 1$ many elements in their union, which is strictly less than $|G|$. So G cannot be equal to the union.

   (c) Suppose we have a transitive action of a finite group G on a finite set X, then we have by part (a) and (b), $\bigcup_{x \in X} G_x = \bigcup_{g \in G} gG_{x_0}g^{-1} \subsetneq G$. Pick an $h \in G \setminus \bigcup_{x \in X} G_x$, then by definition h fixes none of $x \in X$, i.e. $h \cdot x \neq x$ for all $x \in X$. Therefore we have proved the statement.

   Alternatively, we can resort to Burnside's lemma and prove this result without using part (a) and (b). Suppose G acts transitively on X, then there is only one orbit,

$|X/G| = 1$. Assume on the contrary that for all $g \in G$ there is some $x \in X$ being fixed $g \cdot x = x$, in other words $|X_g| \geq 1$. Then by Burnside's lemma,

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{|X_e| + \sum_{g \neq e} |X_g|}{|G|} = \frac{|X| + |G| - 1}{|G|} > 1,$$

which is a contradiction.

5. Let G be the group of all orientation preserving rotations in $\mathbb{R}^3$, i.e. G consists of all 3×3 orthogonal matrices which have determinant 1. This is called the special orthogonal group $G = SO(3) = \{A \in GL(\mathbb{R}^3) : AA^T = I, \det(A) = 1\}$. You can simply think of it as the group of all rotations acting on the unit sphere $\mathbb{S}^2$. This action is transitive because given any two points, you can connect them by a great circle and rotate along the circular arc. However, any non-identity rotation is really just rotating along some axis, hence must fix a pair of antipodal points. Therefore the statement of Q4c does not hold in this case.

   The result of Q4c fails in general because the statement of Q4b no longer holds if we just take an infinite group G with arbitrary subgroup H whose index is infinite. For example, let $U \subset GL(n, \mathbb{C})$ be the subgroup of upper triangular matrices. This is an infinite index subgroup with $GL(n, \mathbb{C}) = \bigcup_{A \in GL(n,\mathbb{C})} AUA^{-1}$ since any matrix is conjugate to an upper triangular matrix. The same phenomenon happens for $SO(3)$ above, the subgroup $H \leq SO(3)$ consisting of rotations along a fixed axis satisfies $SO(3) = \bigcup_{A \in SO(3)} AHA^{-1}$.

6. We can let G acts on the conjugacy class C, by $\rho(g) : c \mapsto gcg^{-1}$. This is a transitive action on a finite set C by definition. Hence by Q4c, there must be some $g \in G$ acting freely on C, in other words $gcg^{-1} \neq c$ for all $c \in C$, i.e. such g does not commute with any $c \in C$.

Here is aheuristic for why Burnside's lemma is true that I described in the tutorial. If $M = \frac{1}{|G|} \sum_{g \in G} |X^g|$ is the average number of x fixed by g, we can determine the average number of g fixing a particular x: since there are $|G| \cdot M$ many fixed points of the group action, on average there are $\frac{|G| \cdot M}{|X|}$ many g fixing an x. Note that this number is also the average size of the stabilizer of x. So we can then determine the average size of the orbit by $|G| / \frac{|G| \cdot M}{|X|} = \frac{|X|}{M}$. Therefore the number of orbits should be $\frac{|X|}{\text{average size of orbit}} = |X| / \frac{M}{|X|} = M$.